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Sustainable ecosystem management using optimal control theory: Part 2 (stochastic systems)

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Abstract

Sustainable development of ecosystems through external ecosystem management is assuming importance for the environmentalists. To that effect, previous work by the authors looked at the option of manipulating population dynamics of the species in an ecosystem to achieve sustainability. Fisher information is used as the quantifying measure of sustainability and optimal control theory is used to derive the control profiles. However, that work considered only deterministic systems. Uncertainty being prevalent in all systems, particularly in natural systems, this paper extends that work to analyse uncertain systems. Predator–prey models are used to model the species populations and different control philosophies are compared. Ito mean reverting process is used to model the stochastic process, and stochastic maximum principle is used to derive the control profiles. The results for the objective of FI variance minimization qualitatively agree with those for the deterministic system, while the results for the FI maximization objective differ. It is observed that the instability associated with the FI maximization objective for deterministic systems is absorbed by the noise introduced by the uncertainty. Quantitatively, it is observed that the degree of uncertainty, along with its presence, is also important to identify the most appropriate management strategy.

Keywords: Sustainability; Ecosystem management; Stochastic optimal control; Uncertainty analysis

1. Introduction

Part 1 of the work (Shastri and Diwekar, 2006) presents the application of sustainability in ecosystem management. It analyses the option of achieving a sustainable ecosystem through external manipulations of the population dynamics of the species in an ecosystem. The work uses Fisher information (FI) based sustainability hypotheses, proposed by Cabezas and Fath (2002) and Fath et al. (2003), to quantify ecosystem sustainability and to formulate two different objectives. These objectives are compared for two different control options, the profiles for which are derived using the optimal control theory. Predator–prey model is used to model the population dynamics in the ecosystem. Comparative observations lead to conclusions about the relative success of each of the objectives and control philosophies. The analysis in part 1 is restricted to deterministic systems, where all the system parameters are known with certainty. However, this is rarely true. Most of the physical systems are non-deterministic, known as uncertain or stochastic, to varied extents. This situation is more severe in natural ecosystems. Hence, consideration of uncertainty is paramount to this analysis.

This second part of the work looks at the task of sustainable ecosystem management under uncertainty. To perform a comparative study with the results presented in part 1 (Shastri and Diwekar, 2006), the same cases after incorporating model uncertainties are considered. This introduces two new aspects to the task at hand: uncertainty modelling and control profile derivation. The mortality rate is considered uncertain, and it is modelled using the Ito mean reverting process (Diwekar, 2005). To derive the

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optimal control profiles, recently proposed stochastic maximum principle (Rico-Ramirez and Diwekar, 2004) is used.

The article is organized as follows. The next section briefly reviews the theoretical basics of this work. While the topics of uncertainty modelling and stochastic optimal control are explained in some details, the other details can be found in part 1 (Shastri and Diwekar, 2006). Section 3 gives the problem specific details, and Sections 4 and 5 report and discuss various results for the three species ecosystem model. The article ends with conclusions in Section 6.

2. Theoretical basics

This work considers the objectives and control philosophies used in Part 1 (Shastri and Diwekar, 2006). The objectives are the maximization of time averaged Fisher Information and minimization of FI variance over time, while the control philosophies are top–down control and bottom–up control. The species populations in the ecosystem are again represented by the three species predator–prey model (tri-trophic food chain model), which is a class of more general food web models. For this analysis though, the mortality rate of the predator (middle level species) is considered uncertain. This consideration of uncertainty calls for changes in the model representation and the optimal control problem formulation. The explanation of these two topics follows.

2.1. Predator–prey model and Ito process

The model used to represent the species interactions in the ecosystem is the Rosenzweig–MacArthur model, a predator–prey model. The model is given by the following set of differential equations:

$$f_1 = \frac{\mathrm{d}x_1}{\mathrm{d}t} = x_1\left[r\left(1 - \frac{x_1}{k}\right) - \left(\frac{a_2 x_2}{b_2 + x_1}\right)\right], \quad (1)$$

$$f_2 = \frac{\mathrm{d}x_2}{\mathrm{d}t} = x_2\left[e_2\frac{a_2 x_1}{b_2 + x_1} - \frac{a_3 x_3}{b_3 + x_2} - x_4\right], \quad (2)$$

$$f_3 = \frac{\mathrm{d}x_3}{\mathrm{d}t} = x_3\left[e_3\frac{a_3 x_2}{b_3 + x_2} - d_3\right], \quad (3)$$

where, x_1, x_2 and x_3 are the population variables of three species in a food chain, in the ascending order of location. These species are referred to in this work as prey (x_1), predator (x_2) and super-predator (x_3). r and K are the prey growth rate and prey carrying capacity, respectively, and a_i, b_i and e_i, $i = 2, 3$, are the maximum predation rate, half saturation constant, and efficiency of the predator ($i = 2$) and super-predator ($i = 3$), respectively. d_3 is the death rate (mortality rate) of super-predator, while x_4 is the death rate of the predator, which is the uncertain variable.

Quite often, probability distribution is used to model the uncertain variables. However, this representation is appropriate to model static uncertainties only. In the current problem though, the predator mortality rate is expected to show time dependent variations, such as seasonal variations in mortality. In such a case, the mortality rate needs to be modelled as a time dependent uncertainty, also known as the stochastic process.

In any stochastic process, there is a variable evolving over time t, and both, time t as well as the variable itself, can assume discrete as well as continuous nature. Wiener process, also known as Brownian motion, is a simple continuous time and continuous state stochastic process. It can be used to model a variety of continuous stochastic processes (Dixit and Pindyck, 1994; Diwekar, 2003). The Wiener process is represented as

$$\mathrm{d}z = \varepsilon_t\sqrt{\mathrm{d}t}, \quad (4)$$

where, $\mathrm{d}z$ is the random variable, and ε_t is a normally distributed random variable, with zero mean and unit standard deviation. Random variable $\mathrm{d}z$ has the property that the expectation is zero ($E[\mathrm{d}z] = 0$) and the variance is $\mathrm{d}t$ ($var[\mathrm{d}z] = \mathrm{d}t$). Using this definition of the Wiener process, the general form of the Ito process is given as

$$\mathrm{d}x = a(x,t)\,\mathrm{d}t + b(x,t)\,\mathrm{d}z. \quad (5)$$

Here, $x(t)$ is the continuous time stochastic variable that is to be modelled, and $\mathrm{d}z$ is the Wiener increment, as defined by Eq. (4). $a(x,t)$ and $b(x,t)$ are known (non-random) functions. $a(x,t)$ is the drift parameter, and $b(x,t)$ is the variance parameter. The first term of the equation constitutes the deterministic part, while the second term constitutes the stochastic part. The mean of the Ito process is $E[\mathrm{d}x] = a(x,t)\,\mathrm{d}t$, and the variance is $var[\mathrm{d}x] = b^2(x,t)\,\mathrm{d}t$.

The general equation of the Ito process (Eq. (5)) is the origin of different stochastic processes, such as the simple Brownian motion with drift, the geometric Brownian motion and the mean reverting Ito process. The geometric Brownian motion is frequently used to model security prices, as well as interest rates, wages rates etc. However, this work uses the mean reverting Ito process to model the predator mortality owing to its success in modelling stochastic variables across seemingly disparate fields, including human mortality (Diwekar, 2005).

The mean reverting Ito process is represented as

$$\mathrm{d}x = \eta\,(\bar{x} - x)\,\mathrm{d}t + \sigma\,\mathrm{d}z. \quad (6)$$

Here, η is the speed of reversion, σ is the constant variance parameter, and $\bar{x}$ is the normal or mean level of x to which $x(t)$ tends to revert. The expected change in x depends on the difference between x and $\bar{x}$. The characteristic of the mean reverting Ito process is that, although it models the random variable fluctuations for a short time, in the long run, the variable is drawn back to the mean value. If the variance rate is growing with x, then the Ito mean reverting process is represented as

$$\mathrm{d}x = \eta\,(\bar{x} - x)\,\mathrm{d}t + \sigma x\,\mathrm{d}z. \quad (7)$$

Table 1
Tri-trophic food chain model: stable parameter set

Prey	Predator	Super-predator	Ito process
$x_1(0) = 100$	$x_2(0) = 75$	$x_3(0) = 150$	$x_4(0) = 1.0$
$r = 1.2$	$a_2 = 2.0$	$a_3 = 0.1$	$\eta = 0.3535$
$K = 710$	$b_2 = 200$	$b_3 = 250$	$\sigma = 0.1205$
	$e_2 = 1.12$	$e_3 = 1.12$	
	$\bar{d_2} = 1.0$	$d_3 = 0.04$	

The mean reverting process has been used to model many stochastic variables, such as crude oil and copper prices (Dixit and Pindyck, 1994), relative volatility of non-ideal mixtures (Ulas and Diwekar, 2004) and also the human mortality rate (Duggempudi and Diwekar, 2003; Diwekar, 2005). In this work, the Ito mean reverting process represented by Eq. (7) is used to model the predator mortality rate. Although, no data is presented to support the validity of such an assumption, its success in modelling the human mortality rate motivated its use in the current work. The equation to model the mortality rate therefore is given as

$$f_{ito} = \frac{dx_4}{dt} = \eta(\bar{x_4} - x_4) + \frac{\sigma\varepsilon}{\sqrt{\Delta t}} x_4, \tag{8}$$

$$f_4 = \eta(\bar{x_4} - x_4). \tag{9}$$

Here, $\bar{x_4}$ is the mean mortality rate and the other symbols have the previously assigned meanings. Eqs. (8) and (9), along with Eqs. (1)–(3), represent the stochastic tri-trophic food chain model. The model and the Ito process parameters are given in Table 1.

2.2. Optimal control theory

As in Part 1 (Shastri and Diwekar, 2006), this work uses the theory of optimal control to derive the top–down and bottom–up control profiles.

For deterministic systems, Pontryagin's maximum principle was used to derive the optimal control equations. However, this method cannot be used in the presence of stochastic processes. For such cases, methods based on the Ito's Lemma need to be used. One such method, the stochastic maximum principle, has been recently proposed (Rico-Ramirez et al., 2003; Rico-Ramirez and Diwekar, 2004), and is used in this work. In this method, the stochastic dynamic programming formulation is converted into a stochastic maximum principle formulation. The main advantage of using this approach is that the solution to the partial differential equations in dynamic programming formulation is avoided. Instead, a set of ordinary differential equations needs to be solved as a boundary value problem. An exhaustive explanation of the theory is beyond the scope of this article, and interested readers are referred to Rico-Ramirez and Diwekar (2004). Given here are the final equations needed to derive the optimal control law.

Consider a system represented by the following set of differential equations:

$$dx = f(x, u, t)\,dt + g\,dz, \tag{10}$$

where, x is the state variable vector of dimension n ($x(t) \in R^n$), and u is the control variable vector of dimension m ($u(t) \in R^m$). The starting condition for the state vector is given by $x(t_0) = x_0$, and the final condition at time T is $x(T)$. Let $1, \ldots, n_k$ be the set of deterministic states, and $n_{k+1}, \ldots, n$ be the set of uncertain states. The second part of Eq. (10) models the uncertainty. For deterministic states, the function $g = 0$. In optimal control, there is a time dependent performance index, which, in this case, is represented as

$$J(t_0) = \int_{t_0}^{T} F(x(t), u(t), t)\,dt \tag{11}$$

where, F is the function to be optimized over the time interval of $[t_0, T]$. The Hamiltonian for this stochastic case is defined as

$$H(x, u, t) = F(x, u, t) + \lambda' f(x, u, t) + \tfrac{1}{2} g^2 w, \tag{12}$$

where, $\lambda(t)$ is the set of costate or adjoint variables ($\lambda(t) \in R^n$) (the first derivatives of the objective function F with respect to state variables), and λ' represents the matrix transpose. $\omega(t)$ represent the second derivatives of the objective function F with respect to the state variables. This term is included due to the Ito process contribution. The optimal control law is then given by the solution of the following set of equations:

State equation

$$\dot{x}_i = \frac{\partial H}{\partial \lambda_i} = f, \quad i = 1, \ldots, n. \tag{13}$$

Costate equation

$$-\dot{\lambda}_i = \frac{\partial H}{\partial x_i} = \frac{\partial f'}{\partial x_i}\lambda_i + \frac{\partial F}{\partial x_i}, \quad i = 1, \ldots, n, \tag{14}$$

$$\frac{dw_j}{dt} = -2w_j \frac{\partial}{\partial x_j} f_j - \frac{1}{2} w_j \frac{\partial^2}{\partial x_j^2}(g_j^2) - \lambda_j \frac{\partial^2}{\partial x_j^2} f_j, \quad j = n_{k+1}, \ldots, n. \tag{15}$$

Stationarity condition

$$0 = \frac{\partial H}{\partial u_p} = \frac{\partial F}{\partial u_p} + \frac{\partial f'}{\partial u_p}\lambda, \quad p = 1, \ldots, m. \tag{16}$$

This is a set of $2n + (n - n_k)$ ordinary differential equations (state and costate equations) and m algebraic equations (stationarity condition), and it is solved as a boundary value problem. The boundary values of the state and costate variables depend on the problem specification, while the boundary values for w are given as $w(T) = 0$ (Rico-Ramirez and Diwekar, 2004). The control trajectory obtained is optimal for the considered objective function and starting conditions.

3. Optimal control problem specification

Based on the background theory explained in the last section, this sections gives the optimal control problem formulations used in this work. The time averaged Fisher information for a system with three species (Tri-trophic model) is given by

$$I_t = \frac{1}{T_c}\int_0^{T_c}\left(\frac{R_2^2}{R_1^3}\right)\mathrm{d}t, \tag{17}$$

where, T_c is the cycle time of the system, and

$$R_1 = \left(\frac{\mathrm{d}x_1}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}x_2}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}x_3}{\mathrm{d}t}\right)^2 = (f_1)^2 + (f_2)^2 + (f_3)^2, \tag{18}$$

$$R_2 = \frac{\mathrm{d}x_1}{\mathrm{d}t}\frac{\mathrm{d}^2x_1}{\mathrm{d}t^2} + \frac{\mathrm{d}x_2}{\mathrm{d}t}\frac{\mathrm{d}^2x_2}{\mathrm{d}t^2} + \frac{\mathrm{d}x_3}{\mathrm{d}t}\frac{\mathrm{d}^2x_3}{\mathrm{d}t^2} = f_1\bar{f}_1 + f_2\bar{f}_2 + f_3\bar{f}_3. \tag{19}$$

Here, $\bar{f}_1$, $\bar{f}_2$ and $\bar{f}_3$ are the second derivatives of x_1, x_2 and x_3 with time, respectively. The objectives are given as

- maximization of Fisher information

$$J = \mathrm{Max}\ \frac{1}{T}\int_0^{T}\left(\frac{R_2^2}{R_1^3}\right)\mathrm{d}t; \tag{20}$$

- minimization of Fisher information variance

$$J = \mathrm{Min}\int_0^{\mathrm{T}}(I_t - I_{constant})^2\,\mathrm{d}t, \tag{21}$$

where, T is the total time horizon under consideration. I_t is given by Eq. (17), and $I_{constant}$ is the value around which the Fisher information variation is to be minimized. The top–down and bottom–up control philosophies are compared by performing separate analyses using d_3, the mortality rate of super-predator x_3, and K, the prey carrying capacity, as the control variables, respectively.

Stochastic maximum principle, as explained in Section 2.2, is followed to develop the control equations. Given below are the general equation forms for the tri-trophic model and the objective of FI maximization. The state equations are given by Eqs. (1)–(3) and (8), while the Hamiltonian is given by

$$H(x,u,t) = F + \lambda_1 f_1 + \lambda_2 f_2 + \lambda_3 f_3 + \lambda_4 f_4 + \tfrac{1}{2}w\sigma^2 x_4^2. \tag{22}$$

In this case, F is given as

$$F = \frac{1}{T}\left(\frac{R_2^2}{R_1^3}\right). \tag{23}$$

The adjoint equations are given as

$$-\dot{\lambda}_i = \frac{\partial H}{\partial x_i}, \quad t \leqslant T \quad \text{and} \quad i = 1,2,3,4, \tag{24}$$

$$\frac{\mathrm{d}w}{\mathrm{d}t} = -2w\frac{\partial}{\partial x_4}f_4 - \frac{1}{2}w\frac{\partial^2}{\partial x_4^2}(\sigma^2 x_4^2) - \lambda_4\frac{\partial^2}{\partial x_4^2}f_4. \tag{25}$$

Here,

$$\frac{\partial H}{\partial x_i} = \lambda_1\frac{\partial f_1}{\partial x_i} + \lambda_2\frac{\partial f_2}{\partial x_i} + \lambda_3\frac{\partial f_3}{\partial x_i} + \lambda_4\frac{\partial f_4}{\partial x_i} + \frac{1}{2}w\frac{\partial}{\partial x_i}(\sigma^2 x_4^2) + \frac{\partial F}{\partial x_i}, \tag{26}$$

$$\frac{\partial F}{\partial x_i} = \frac{1}{T}\frac{R_2}{R_1^3}\left[2\frac{\partial R_2}{\partial x_i} - 3\frac{R_2}{R_1}\frac{\partial R_1}{\partial x_i}\right], \tag{27}$$

$$\frac{\partial R_1}{\partial x_i} = 2f_1\frac{\partial f_1}{\partial x_i} + 2f_2\frac{\partial f_2}{\partial x_i} + 2f_3\frac{\partial f_3}{\partial x_i}, \tag{28}$$

$$\frac{\partial R_2}{\partial x_i} = f_1\frac{\partial \bar{f}_1}{\partial x_i} + \bar{f}_1\frac{\partial f_1}{\partial x_i} + f_2\frac{\partial \bar{f}_2}{\partial x_i} + \bar{f}_2\frac{\partial f_2}{\partial x_i} + f_3\frac{\partial \bar{f}_3}{\partial x_i} + \bar{f}_3\frac{\partial f_3}{\partial x_i}, \tag{29}$$

where, $i = 1,2,3,4$. The optimality/stationarity condition is given by Eq. (30).

$$\frac{\partial H}{\partial u} = 0. \tag{30}$$

Here,

$$\frac{\partial H}{\partial u} = \lambda_1\frac{\partial f_1}{\partial u} + \lambda_2\frac{\partial f_2}{\partial u} + \lambda_3\frac{\partial f_3}{\partial u} + \lambda_4\frac{\partial f_4}{\partial u} + \frac{1}{2}w\frac{\partial}{\partial u}(\sigma^2 x_4^2) + \frac{\partial F}{\partial u}, \tag{31}$$

$$\frac{\partial F}{\partial u} = \frac{1}{T}\frac{R_2}{R_1^3}\left[2\frac{\partial R_2}{\partial u} - 3\frac{R_2}{R_1}\frac{\partial R_1}{\partial u}\right], \tag{32}$$

$$\frac{\partial R_1}{\partial u} = 2f_1\frac{\partial f_1}{\partial u} + 2f_2\frac{\partial f_2}{\partial u} + 2f_3\frac{\partial f_3}{\partial u}, \tag{33}$$

$$\frac{\partial R_2}{\partial u} = f_1\frac{\partial \bar{f}_1}{\partial u} + \bar{f}_1\frac{\partial f_1}{\partial u} + f_2\frac{\partial \bar{f}_2}{\partial u} + \bar{f}_2\frac{\partial f_2}{\partial u} + f_3\frac{\partial \bar{f}_3}{\partial u} + \bar{f}_3\frac{\partial f_3}{\partial u}, \tag{34}$$

where, u can either be K or d_3, depending on the control philosophy employed. Eq. (30) gives an implicit relationship for the control variable u as a function of states x_1, x_2, x_3 and x_4 and the model parameters. Owing to the complexity of the equation, an explicit relationship for u cannot be derived. Starting conditions of the system (populations at the start), $x_1(0)$, $x_2(0)$, $x_3(0)$ and $x_4(0)$, are known, and since the final states are free, adjoint variables at the final time are known, i.e. $\lambda_1(T) = 0$, $\lambda_2(T) = 0$, $\lambda_3(T) = 0$ and $\lambda_4(T) = 0$, while $w(T) = 0$. The final state of the system is not constrained, and the objective does not contain final time function. The time horizon for the control problem is considered to be large enough, so that the control law is only state dependent (Kirk, 1970). Numerical technique of the steepest ascent of Hamiltonian is used to solve the boundary value problem

(Kirk, 1970; Diwekar, 1996). To minimize the effect of random normal sampling of ε_t, the Ito process is modelled for smaller time steps and mortality rate values at the desired time steps are used in the optimal control problem.

The next section gives the simulation results for the tri-trophic food chain model.

4. Results and discussion: three species predator–prey model

The analysis compares both control options and both objectives on the tri-trophic food chain model given by Eqs. (1)–(3) and (8). Moreover, the uncontrolled model is considered to have undesirable dynamics. It simulates the situations when the ecosystem needs external intervention to avoid imbalance. The objective is to recover the system from the disturbance in a sustainable manner, i.e. achieving dynamic stability. The analysis assesses the ability of different sustainability based objectives and control philosophies to perform this task. Three such cases are considered:

Case A: Excessive prey and predator variations—Populations x_1 and x_2 vary excessively.

Case B: Super-predator extinction—Population x_3 is going extinct.

Case C: Super-predator explosion—Population x_3 is exploding.

These cases are simulated by modifying the model parameters reported in Table 1. When K or d_3 is the control variable, reported value is the starting guess for the steepest ascent algorithm. The control variables are constrained to avoid numerical problems. The constraints on control variables are: $0.03 \leqslant d_3 \leqslant 0.05$ for the top–down control and $500 \leqslant K \leqslant 900$ for the bottom–up control.

The Ito process parameter, ε_t, is first sampled and stored for the subsequent simulations. The random mortality rate variation for the given samples is shown in Fig. 1. The model is then simulated for the uncontrolled case, and values of the average Fisher information and Fisher information standard deviation are noted. The model is then subjected to the two different control philosophies. The results for the three cases follow.

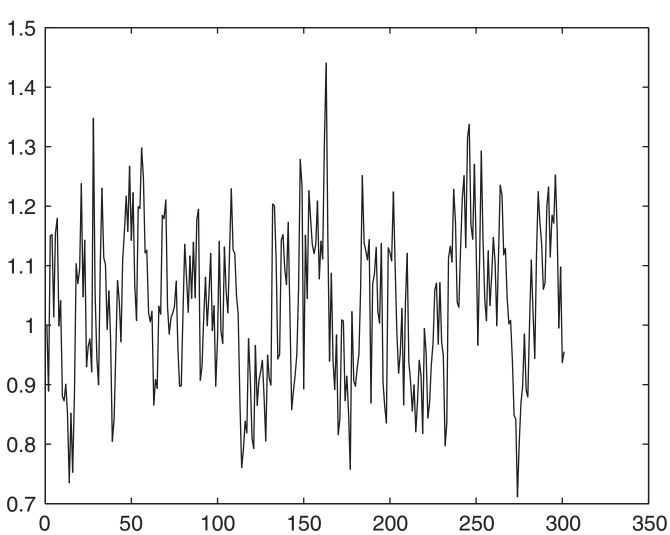

Fig. 1. Time dependent random predator mortality rate.

Table 2
Tri-trophic food chain model: parameter set for case A

Prey	Predator	Super-predator	Ito process
$x_1(0) = 100$	$x_2(0) = 75$	$x_3(0) = 150$	$x_4(0) = 1.0$
$r = 1.2$	$a_2 = 2.0$	$a_3 = 0.1$	$\eta = 0.3535$
$K = 710$	$b_2 = 227.27$	$b_3 = 250$	$\sigma = 0.1205$
	$e_2 = 1.35$	$e_3 = 1.29$	
	$\bar{d}_2 = 1.0$	$d_3 = 0.04$	

4.1. *Case A: excessive prey and predator variations*

The model parameters used to simulate this case are reported in Table 2. Although the uncontrolled system is not dynamically unstable, the increased variation in prey and predator populations is undesirable. This is because as these populations decrease, they are in a greater danger of becoming extinct due to unpredicted events, such as natural disasters, external species invasion etc.

Numerical values for the results are reported in Table 3, which show that all the control options achieve the desired objectives to different extents. The predator–prey dynamics for these results are shown in Fig. 2 for top–down control and Fig. 3 for bottom–up control. Although the plots are somewhat complicated to analyse, careful analysis shows that the predator–prey variations are not affected much by the top–down control (Fig. 2). In comparison, bottom–up control affects the predator–prey population dynamics more severely (Fig. 3), and it is able to reduce the population variations. The result for the FI variance minimization objective is particularly good, since the reduction in variations is maximum for this case. Since the sustainability hypotheses argue that a system with few preferred states is more sustainable, the controlled system is also more sustainable than the uncontrolled one. The super-predator dynamics for these cases are plotted in Fig. 4. The favorable results of variation reduction are accompanied by an increase in the super-predator population. However, a substantial rise in the super-predator population for FI variance minimization using top–down control does not significantly affect the predator–prey dynamics. This observation indicates that the variations in the super-predator population are the effect and not the cause of reducing predator–prey variations. The control variable profiles for this case are given in Figs. 5 and 6. They show that control variables fluctuate more rapidly (at higher frequency) for FI maximization objective than for FI variance minimization objective. Since the FI variance minimization objective minimizes the variation between the average FI of each model cycle, the piecewise nature of the control action is evident in these plots, particularly for the top–down control. The results for FI variance

Table 3
Results for case A: excessive prey and predator population variation

Type of analysis	Top–down control		Bottom–up control	
	FI	FI standard deviation	FI	FI standard deviation
Uncontrolled model	5.76×10^{-5}	4.24×10^{-5}	5.76×10^{-5}	4.24×10^{-5}
FI maximization	5.91×10^{-5}	2.45×10^{-5}	2.13×10^{-3}	9.30×10^{-3}
FI variance minimization	4.34×10^{-5}	1.89×10^{-5}	2.95×10^{-5}	7.75×10^{-6}

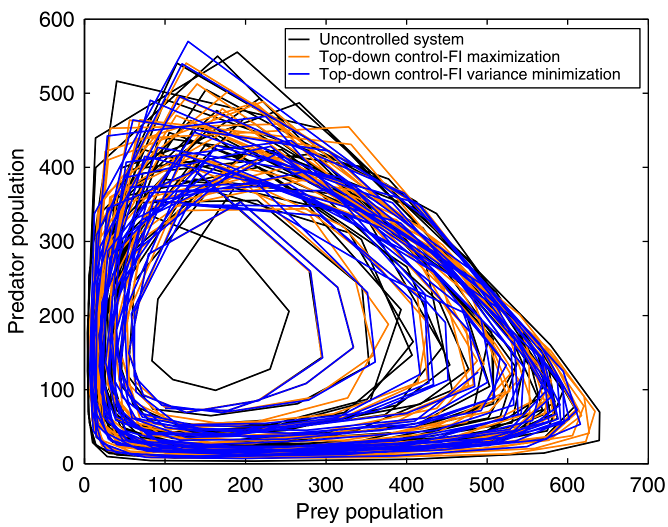

Fig. 2. Predator–prey population dynamics for case A: top–down control.

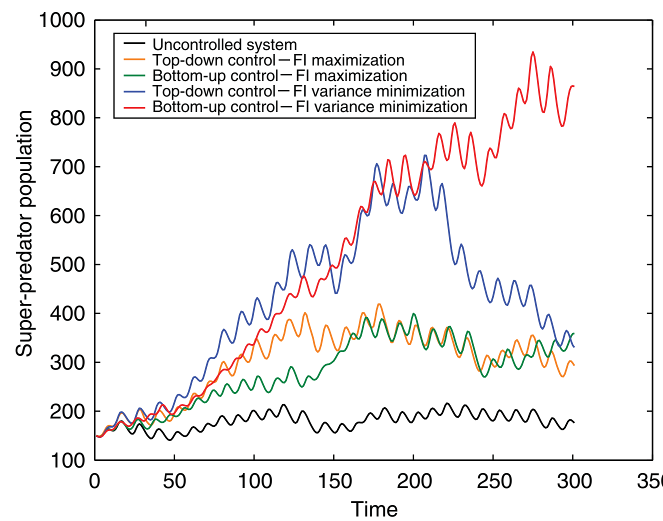

Fig. 4. Super-predator population dynamics for case A.

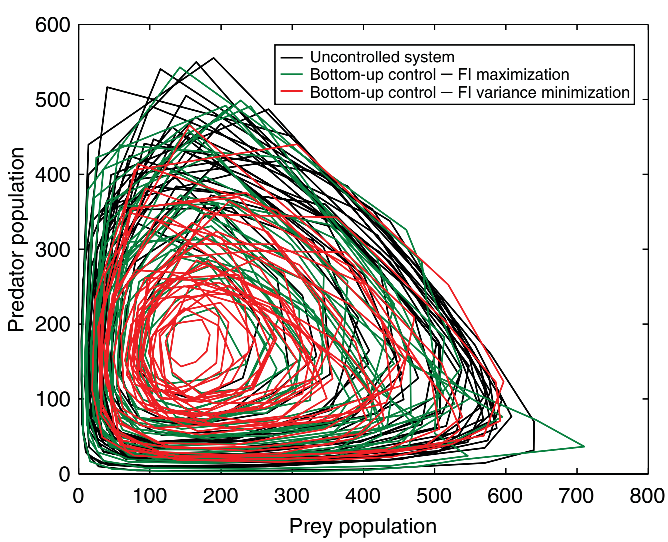

Fig. 3. Predator–prey population dynamics for case A: bottom–up control.

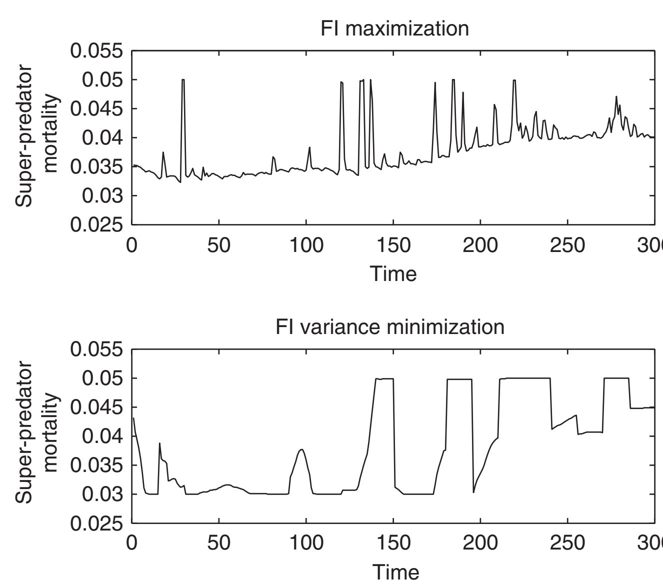

Fig. 5. Top–down control variable profile for case A.

minimization will, therefore, be easier to implement on an actual system, since the changes will be less frequent.

A comparison with the deterministic model (Part 1) (Shastri and Diwekar, 2006) indicates that the results are qualitatively similar. For both the models, the bottom–up control achieves better reduction in the population variation of prey and predators.

4.2. Case B: super-predator extinction

To simulate super-predator extinction, predator half saturation constant is taken as $b_2 = 181.81$. The numerical results obtained for this case are reported in Table 4, which again indicate that the desired objective is achieved in terms of numerical values. The super-predator population dynamics for this case are plotted in Fig. 7. The plots show that all the control options are able to elevate the super-predator population above that for the uncontrolled case. The degree of success in restricting the super-predator extinction varies for different control options. The top–-down control is able to exercise a stronger control on the super-predator population. For both objectives, it results in a substantial rise in the super-predator population. Continued rise in the population can result in an unstable system. The control by bottom–up control is slower, particularly for the objective of FI variance minimization. Simulations of the system for a longer time duration show that bottom–up control is not able to elevate the super-predator population back to the initial level, but manages the avoid super-predator extinction. Since species extinction is avoided in all cases, they represent a more sustainable system as compared to the uncontrolled one. The predator–prey dynamics for this case are quite mixed up. But careful consideration indicates that the bottom–up control impacts the predator–prey dynamics more than the top–down control (plots not shown). The control variable profiles for the top–down and bottom–up control are shown in Figs. 8 and 9, respectively. Since the super-predator population can be increased by reducing its mortality rate, the mortality rate control variable is maintained around the lower bound 0.03. The bottom–up control variable varies over a greater range. Since its effect on the super-predator population is indirect, a direct interpretation is difficult. These profiles reinforce the point that FI maximization objective gives control variable profiles that are relatively difficult to implement, due to more rapid fluctuations.

A comparison with the deterministic model (part 1) (Shastri and Diwekar, 2006) indicates that the results are qualitatively quite similar. For both control options, the FI maximization objective affects the super-predator population more strongly than the FI variance minimization objective. Also, bottom–up control effects on the prey and predator populations are stronger than top–down control. For both the models the super-predator response seems to be sensitive to the objective (FI maximization).

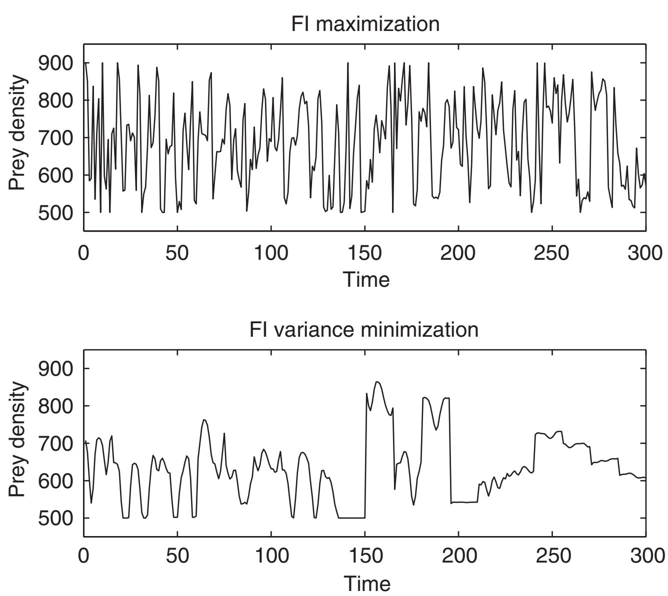

Fig. 6. Bottom–up control variable profile for case A.

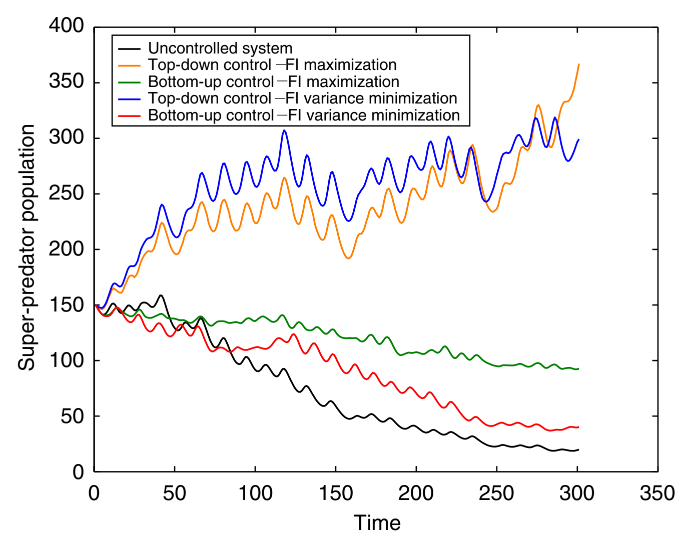

Fig. 7. Super-predator population dynamics for case B.

Table 4
Results for case B: super-predator extinction

Type of analysis	Top–down control		Bottom–up control	
	FI	FI standard deviation	FI	FI standard deviation
Uncontrolled model	1.39×10^{-4}	1.08×10^{-4}	1.39×10^{-4}	1.08×10^{-4}
FI maximization	1.85×10^{-4}	2.25×10^{-4}	8.08×10^{-1}	3.61
FI variance minimization	1.21×10^{-4}	9.54×10^{-5}	6.58×10^{-5}	2.22×10^{-5}

4.3. Case C: super-predator explosion

To simulate super-predator explosion, $b_2 = 208.33$. The numerical values obtained in this case are reported in Table 5, while the super-predator population dynamics are plotted in Fig. 10. The plots indicate that all the control options are able to restrict the super-predator explosion to different extent. The objective of FI maximization with bottom–up control reduces the population excessively, however, it does not appear to take the super-predator close to extinction. For the other cases, the desired goal of controlling the super-predator extinction is achieved satisfactorily. Regarding the predator–prey dynamics, as in case B, the dynamics are mixed up, but indicate that the bottom–up control affects the predator–prey dynamics more than the top–down control (plots not shown). The control variable profiles for top–down and bottom–up control are shown in Figs. 11 and 12, respectively. The plots, as in the previous cases, show the piecewise nature for FI variance minimization objective. They also emphasize that objective of FI variance minimization results in control variable profiles that will be easier to implement on a physical system.

A comparison with the deterministic model (part 1) (Shastri and Diwekar, 2006) shows that the behavior is qualitatively similar for FI variance minimization objective. However, it is different for the FI maximization objective. For the deterministic system, FI maximization objective resulted in a further rise in the super-predator population, while for the stochastic system, the super-predator

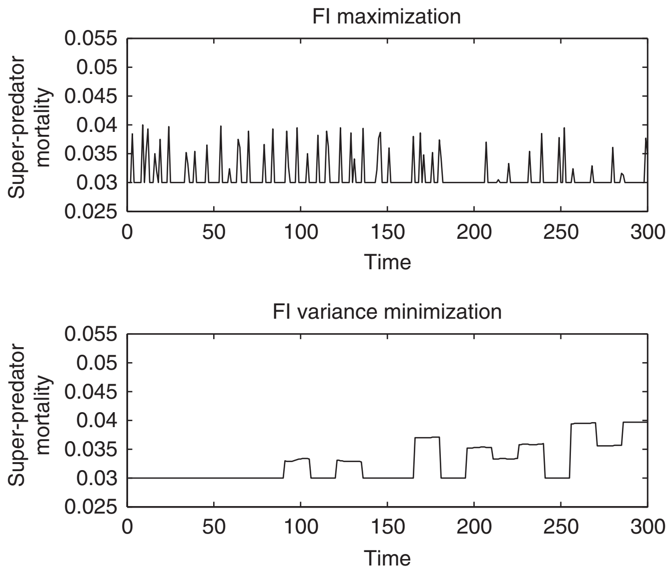

Fig. 8. Top–down control variable profile of case B.

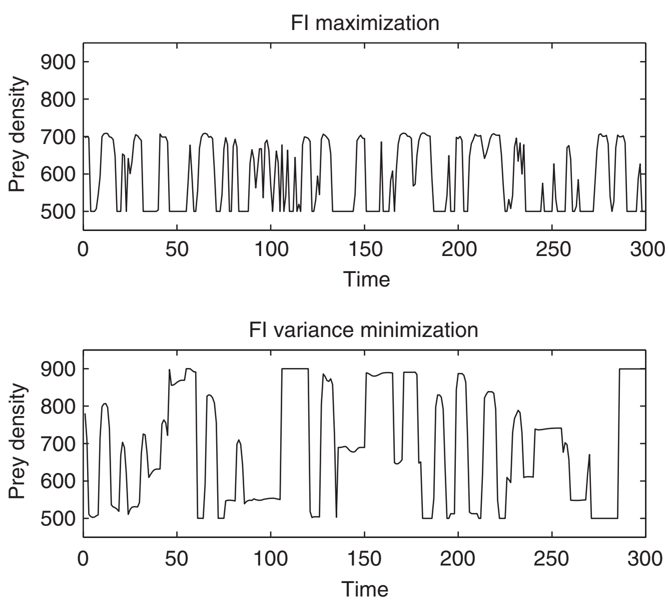

Fig. 9. Bottom–up control variable profile of case B.

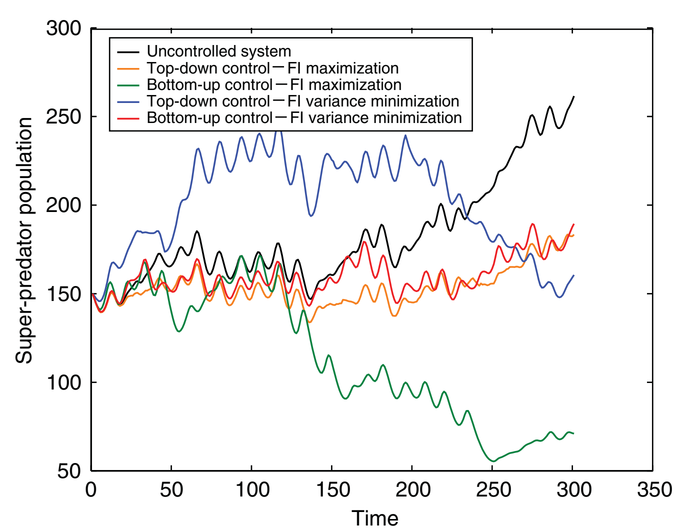

Fig. 10. Super-predator population dynamics for case C.

Table 5
Results for case C: super-predator explosion

Type of analysis	Top–down control		Bottom–up control	
	FI	FI standard deviation	FI	FI standard deviation
Uncontrolled model	1.80×10^{-3}	5.80×10^{-3}	1.80×10^{-3}	5.80×10^{-3}
FI maximization	1.40×10^{-1}	6.26×10^{-1}	1.58	7.07
FI variance minimization	8.79×10^{-5}	5.48×10^{-5}	6.22×10^{-5}	3.00×10^{-5}

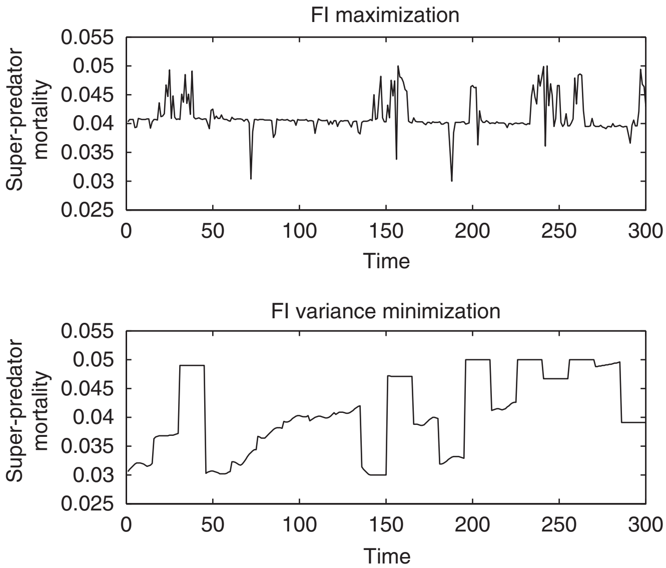

Fig. 11. Top–down control variable profile of case C.

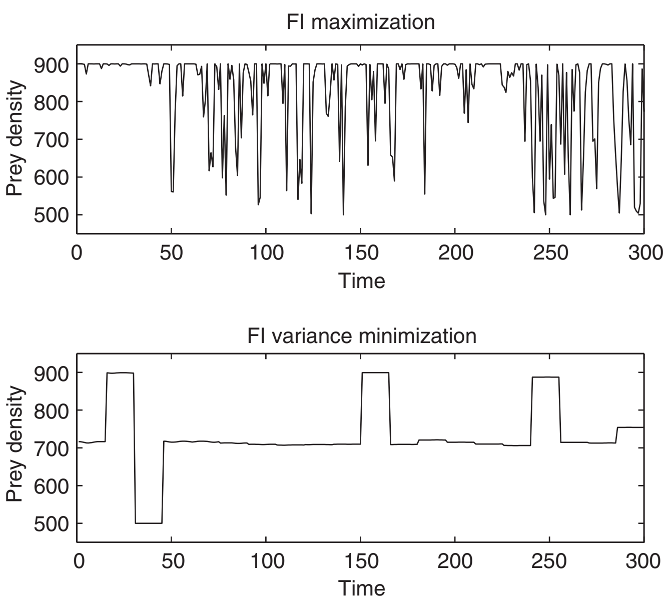

Fig. 12. Bottom–up control variable profile of case C.

population is satisfactorily controlled. This suggests that the noise associated with the uncertainty in this case reduces the destabilizing effect of the FI maximization objective observed for the deterministic systems. The result shows that it might be advantageous to use the objective of FI maximization.

5. Uncertainty impact analysis

Understanding the impact of the extent or degree of uncertainty on the stochastic system dynamics and control is important. This is done here by performing two different studies for the stochastic models:

- analysing the dependence on the Ito process;
- analysing the impact of using stochastic control theory to derive control variable profiles.

Both these cases are analysed for case B of the tri-trophic food chain model (super-predator extinction).

5.1. Ito process dependence

The dependence of the results on Ito process parameters assumes importance, because predator mortality rate representation using the Ito process is an assumption, derived from its successful use to model human mortality. It is quite likely that, although represented by an Ito process, the exact characteristics (parameters) for predator mortality rate are different. This work analyses the dependence on the constant variance parameter σ of the Ito process. The variance parameter for the original Ito process, $\sigma = 0.1205$, is increased to $\sigma = 0.2205$, making the predator mortality rate more variable, as shown in Fig. 13. Case B (super-predator extinction) is analyzed for the new Ito process, where the aim is to avoid super-predator extinction. The super-predator dynamics for this case are shown in Fig. 14. Compared to the case B dynamics presented before (Fig. 7), the uncontrolled super-predator population falls more drastically in this case, due to the increased variance of the Ito process. For the controlled cases, it is seen that the bottom–up control is not able to recover the super-predator population. The interesting aspect though is in the response of the top–down controlled systems, which, for both the objectives, controls the super-predator extinction. With the constraint $0.03 \leqslant d_3 \leqslant 0.05$ on super-predator mortality rate (top–down control variable), the FI maximization objective performed worse than FI variance minimization objective. This is in contradiction to

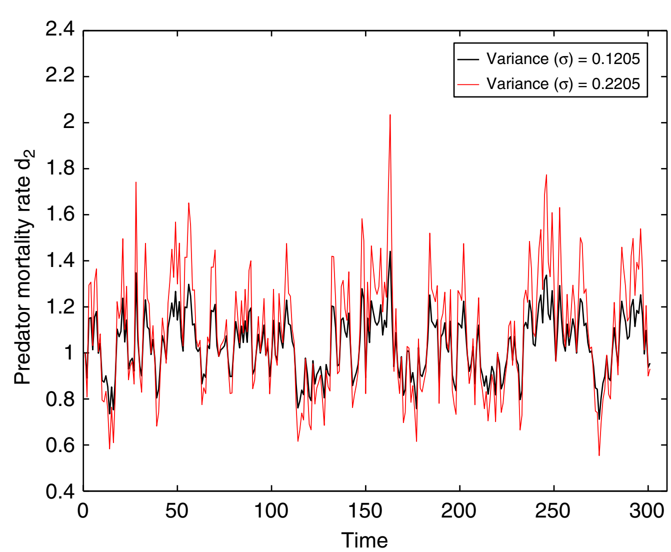

Fig. 13. Comparison of the Ito processes for difference variance parameter.

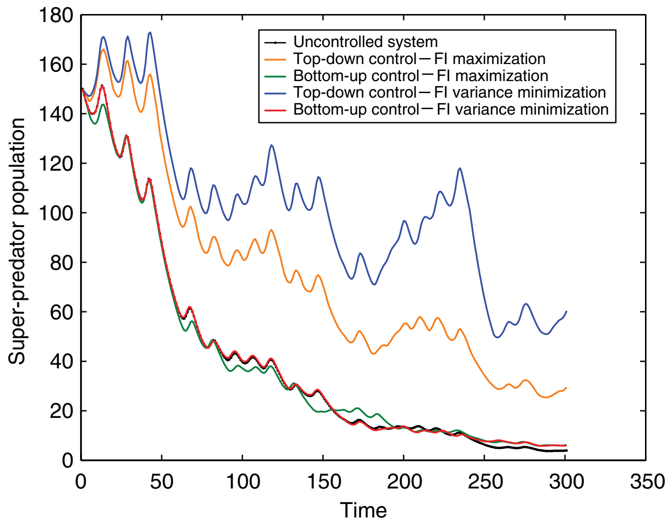

Fig. 14. Super-predator population dynamics for case B and higher Ito process variance.

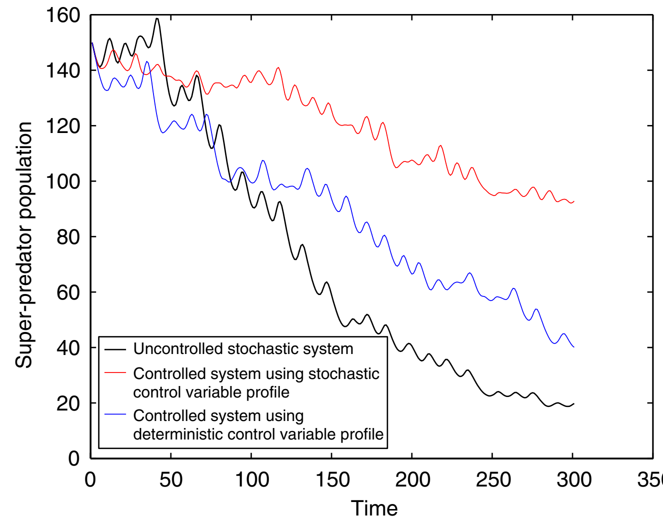

Fig. 15. Comparison of deterministic and stochastic control for model exhibiting super-predator extinction (bottom–up control and FI maximization objective).

the previous result for the extinction case, including that for the deterministic system, suggesting that FI variance minimization might be a better objective overall. A general conclusion is that, under severe disturbances in system dynamics, bottom–up control is not strong enough to control the system.

5.2. *Effect of stochastic optimal control theory*

The advantage of using stochastic optimal control theory to derive the optimal control variable profile is analysed here. Fig. 15 plots the response of the stochastic tri-trophic food chain model exhibiting super-predator extinction (case B), when controlled by prey density (bottom–up control) using FI maximization objective. The following two cases are considered:

- stochastic system controlled by control variable profile generated using stochastic optimal control theory (stochastic maximum principle) as explained in this work;
- stochastic system controlled by control variable profile generated by using deterministic control theory i.e. not considering uncertainty to derive control profile.

The plots indicate that super-predator population is elevated much more using the control profile generated by stochastic optimal control theory. Although, using the control profile generated by the deterministic optimal control theory also restricts super-predator extinction, its performance is clearly inferior to the stochastic control variable profile. One can, therefore, conclude that the stochastic control gives a better result as compared to the deterministic control. This trend is observed, to a greater or lesser extent, for other cases too, highlighting the importance of uncertainty incorporation in control problem solution.

6. Conclusion

The topic of sustainable ecosystem management using external control for deterministic systems has been analysed in the first part of the work (Shastri and Diwekar, 2006). This work attempted to understand the effect of uncertainty on the problem and its solutions. To that effect, analysis of the same (deterministic) system with the incorporation of uncertainty in the mortality rate of the predator species was performed. The uncertain variable was modelled as a mean reverting Ito process, and stochastic maximum principle was employed to solve the optimal control problem.

The results and comparison with the deterministic model simulations, reported in Sections 4 and 5, suggest that the results for the deterministic and stochastic models are qualitatively similar for the objective of FI variance minimization. The results differ for the FI maximization objective. For deterministic systems, this objective results in instability. However, for the stochastic systems, the objective resulted in stable systems. The noise associated with the stochastic model is able to take care of the instability, and hence, FI maximization can be used as an objective. A quantitative comparison indicates that uncertainty impacts the relative extent of success or failure of a management philosophy. It is also shown that not only the presence of uncertainty, but also the degree of uncertainty, is important to rank various management options.

Thus, to summarize, qualitative similarity with the deterministic results confirms the hypothesis that FI variance minimization objective is guaranteed to give a stable response. The FI maximization objective has a stronger impact on the population dynamics of the super-predator, which might also result in unstable systems. In this light, the FI maximization objective involves risk of instability, while FI variance minimization might be viewed as the 'safety first' objective. The impact of bottom–up control on the prey and predator population is stronger than that of top–down control. It is also important to realize that the ignorance of uncertainty can be dangerous for ecosystem management. Further work, particularly analyzing the various management options for different systems and for various situations, such as predator or prey extinction, will help in generalizing the findings presented in this work.

Acknowledgements

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